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Positive sectional curvature, symmetry and Chern's conjecture

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ABSTRACT

Let M be a n-manifold of positive sectional curvature. Suppose that M admits an isometrical torus T^k-action with $k > \frac{n+1}{8} + 1$. The main results of the paper are: (1) the fundamental group $\pi_1(M)$ contains no $\mathbb{Z}_p \oplus \mathbb{Z}_p$ subgroup with p prime and $p \neq 3$ (a partial positive answer to Chern's conjecture); (2) the 2-order element of $\pi_1(M)$ belongs to the center of $\pi_1(M)$.

0. Introduction

Let M be a closed n-manifold of positive sectional curvature. An interest problem, of long history, is to research the fundamental group, $\pi_1(M)$. According to the Meyers theorem [3], $\pi_1(M)$ is finite; and the Synge theorem asserts that if n is even, then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}_2$. Hence, we always (especially in this paper) assume that the dimension of M, n, is odd when we study $\pi_1(M)$.

A famous conjecture [20], known as Chern's conjecture, on the problem above is that any abelian subgroup of $\pi_1(M)$ is cyclic. Note that the conjecture is true if M is a space form of positive constant curvature [19] (even if the universal covering space of M is homeomorphic to S^n with no constrain on the curvature [2]). Due to the work by Shankar ([16], $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ can be realized as a subgroup of $\pi_1(M)$, cf. [1]), we know that the conjecture is false in general cases. However, under certain conditions, there exist some partial positive answers to the conjecture. For example, Rong gets a result as follows:

Theorem 0.1. *(See [11].) Let M be a closed n-manifold of positive sectional curvature. If M admits an isometric circle T^1-action, then $\pi_1(M)$ contains no $\mathbb{Z}_p \oplus \mathbb{Z}_p$ subgroup with $p > \omega(n)$, where $\omega(n)$ is a constant depending on the dimension n.*

In the present paper, we obtain another partial positive answer to the Chern's conjecture, which is stated as follows.

Theorem A. *Let M be a closed n-manifold of positive sectional curvature on which a torus T^k acts effectively by isometries. If $n \geqslant 23$ and if $k > \frac{n+1}{8} + 1$, then $\pi_1(M)$ contains no $\mathbb{Z}_p \oplus \mathbb{Z}_p$ subgroup with p prime and $p \neq 3$.*

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Remark 0.2. Theorem A is just along the proposal by Grove to study firstly closed positively curved manifolds which admit isometrical torus T^k-actions [5,7,9,12,13,18]. When M, a closed n-manifold of positive sectional curvature, admits an isometrical T^k-action, we have the following classifications on $\pi_1(M)$:

(0.2.1) If $k > \frac{n+1}{4}$, then $\pi_1(M)$ is cyclic ([13], cf. [12,18]);
(0.2.2) If $k > \frac{n}{6} + 1$ and $n \neq 11, 15$, then $\pi_1(M) \cong \pi_1(S)$ [6,14];
(0.2.3) If $k > \frac{n}{6}$ and $n \geqslant 25$, then $\pi_1(M) \cong \pi_1(S)$, or $\pi_1(M) \cong \pi_1(L)$ and $n \equiv 5 \bmod 6$ [15],

where S is a 3-dimensional space form of constant curvature one and L is a rational homology sphere of dimension 5. According to (0.2.1) and (0.2.2), it obviously follows that $\pi_1(M)$ contains no $\mathbb{Z}_p \oplus \mathbb{Z}_p$ subgroup with p prime if $k > \frac{n}{6} + 1$ and $n \neq 11, 15$ (cf. Theorem A).

Remark 0.3. So far, we have not find an example, which satisfies the condition of Theorem A, having a fundamental group containing a $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ subgroup. The reason for excluding the case '$p = 3$' in Theorem A is just that the argument in our proof does not work for '$p = 3$' (see the second line from the bottom of the last paragraph of the proof of Lemma 1.5 in Section 2).

Compared to the Chern's conjecture, another interesting property is that any 2-order element of $\pi_1(M)$ belongs to the center of $\pi_1(M)$ if the universal covering space of M is homeomorphic to S^n (even if there is no constrain on the curvature, [2]). The example by Shankar [16] also implies that the property is not true for general closed positively curved manifold.

Inspired by Theorem A, we get the following result in the present paper.

Theorem B. *Let M be a closed n-manifold of positive sectional curvature on which a torus T^k acts effectively by isometries. If $n \geqslant 23$ and if $k > \frac{n+1}{8} + 1$, then*

(B1) *Any 2-order element of $\pi_1(M)$ belongs to the center of $\pi_1(M)$;*
(B2) *If $\pi_1(M)$ is a group of even order or if the T^k-fixed point set is empty, then $\pi_1(M) \cong \pi_1(S)$, where S is a 3-dimensional space form of constant curvature one or a rational homology 5-sphere.*

Remark 0.4. Together with Theorem A, the conclusion (B1) in Theorem B implies that $\pi_1(M)$ contains at most one 2-order element if the closed positively curved n-manifold M with $n \geqslant 23$ admits an isometrical T^k-action with $k > \frac{n+1}{8} + 1$.

Remark 0.5. In (B2) (also in (0.2.2) and (0.2.3)), to prove that $\pi_1(M) \cong \pi_1(S)$ with S being a 3-dimensional space form of constant curvature one or a rational homology 5-sphere, we will verify that $\pi_1(M) \cong \pi_1(K^3)$, where K^3 is a closed 3-manifold of positive sectional curvature [8], or verify that $\pi_1(M) \cong \pi_1(L^5)$, where L^5 is a closed 5-manifold of positive sectional curvature whose universal covering space is a rational homology sphere [2].

The rest of the paper is organized as follows: In Section 1, we will give the proofs of Theorems A and B by assuming two key lemmas. In Section 2, we will give the proof of Lemma 1.5 which is used to prove Theorem A. In Section 3, we will prove Lemma 1.6 used to prove Theorem B. In Appendix A, we will prove an algebraic lemma which implies Lemma 1.1.

1. Proofs of Theorems A and B

In this section, we will give the proofs of Theorems A and B by assuming Lemmas 1.5 and 1.6.

We first observe that under the conditions in Theorem A or B, we can find two involutions of T^k with fixed point sets of dimension $> \frac{n}{2}$ (cf. [6,14,15]).

Lemma 1.1. *Under the assumptions in Theorem A (or Theorem B), there exist at least two involutions $\iota_i \in T^k$ with fixed point components F_i of odd dimension $> \frac{n}{2}$ (note that n is odd). Moreover, the effective part of $T^k|_{F_i}$ has a rank at least $k - 1$.*

Recall that *if M, a closed manifold of positive sectional curvature, admits an isometrical T^k-action, then there exists a circle T^k-orbit $\mathcal{O}$* [7,12,17]. Note that the isotropy group of $\mathcal{O}$ contains a T^{k-1} subgroup. By analyzing the representation of $\mathbb{Z}_2^{k-1} \subset T^{k-1}$ on the normal space of $\mathcal{O}$, we can find the involutions in Lemma 1.1 by Lemma 4.1 in Appendix A. On the other hand, when a torus T^k acts on a closed manifold by isometries, any minimal isotropy group (which does not belong to other isotropy groups) is a finite cyclic subgroup or a T^1-subgroup (Ref. [13]). Hence in Lemma 1.1, we can select F_i such that the effective part of $T^k|_{F_i}$ contains a torus of dimension at least $k - 1$.

Noting that F_i in Lemma 1.1 are totally geodesic submanifolds, we can apply the following powerful tool by Wilking (a basic tool for the paper).

Theorem 1.2. *(See [18].) Let M be a closed n-manifold of positive sectional curvature, and let $N_i \subset M$ be two compact totally geodesic embedded n_i-submanifolds. Then*

(1.2.1) *The inclusion map $N_i \hookrightarrow M$ is $(2n_i - n + 1)$-connected.*
(1.2.2) *If a Lie group G acts on M effectively by isometries, and if N_i is fixed pointwisely by G, then $N_i \hookrightarrow M$ is $(2n_i - n + 1 + \dim(G))$-connected.*
(1.2.3) *If $n_1 + n_2 \geqslant n$ and $n_1 \geqslant n_2$, then the intersection $N_1 \cap N_2$ is totally geodesic and the inclusion $N_1 \cap N_2 \hookrightarrow N_2$ is $(n_1 + n_2 - n)$-connected.*

Recall that the inclusion map $N_i \hookrightarrow M$ is called *h-connected*, if the homotopy groups $\pi_j(M, N_i) = 0$ for $0 \leqslant j \leqslant h$. A special case is that $\pi_1(M) \cong \pi_1(N_i)$ if $h \geqslant 2$. Hence in Lemma 1.1, $\pi_1(M) \cong \pi_1(F_i)$ (note that we assume that n is odd).

Remark 1.3. In (1.2.3), if N_1 and N_2 are fixed pointwisely by G, then we can use the argument of the proof of (1.2.2) [18] to prove that $N_1 \cap N_2 \neq \emptyset$ if $n_1 + n_2 + \dim(G) \geqslant n$. Another comment on (1.2.3) is that $N_1 \cap N_2$ is connected if $n_1 + n_2 \geqslant n + 1$ ([15], cf. [4]).

Now we give three lemmas (Theorem 1.2 plays crucial role in their proofs) which enable us to apply induction to prove Theorems A and B.

Lemma 1.4. *(See [6].) In Theorem 1.2, if $\dim(N_1) = n - 2$ and if $n \geqslant 5$, then $\pi_1(M)$ is cyclic.*

Lemma 1.5. *For $i = 1$ or 2, in Lemma 1.1, if $\dim(F_i) = n - 4$ or $n - 6$ and if F_i is fixed by a T^1-subgroup, then the conclusion in Theorem A is true.*

Lemma 1.6. *For $i = 1$ or 2, in Lemma 1.1, if $\dim(F_i) = n - 4$ or $n - 6$ and if F_i is fixed by a T^1-subgroup, then the conclusions in Theorem B are true.*

Lemmas 1.5 and 1.6 will be proved in Sections 2 and 3, respectively. Now we give the proofs of Theorems A and B by assuming Lemmas 1.5 and 1.6.

Proof of Theorems A and B. By Lemma 1.1, we can find an involution with fixed point component F of dimension $> \frac{n}{2}$, and the effective part of $T^k|_F$ has dimension $\geqslant (k-1)$ (i.e. F admits an isometrical T^{k-1}-action). Note that $\pi_1(M) \cong \pi_1(F)$ because $F \hookrightarrow M$ is at least 2-connected by (1.2.1) (note that we assume that n is odd).

If $\dim(F) = n - 2$, then $\pi_1(M)$ is cyclic by Lemma 1.4.

Assume that $\dim(F) = n - 4$ or $n - 6$. If the effective part of $T^k|_F$ has dimension $k - 1$, i.e. F is fixed by a T^1-subgroup, then we can conclude Theorems A and B by Lemmas 1.5 and 1.6, respectively. If the effective part of $T^k|_F$ also has dimension k, then we apply inductive assumption on F and repeat the above progress if $\dim(F) \geqslant 23$, or apply (0.2.1) and (0.2.2) to complete the proof if $\dim(F) \leqslant 21$.

Assume that $\dim(F) \leqslant 21 \leqslant n - 8$. If $n \geqslant 31$, then $17 \leqslant \dim(F) \leqslant 21$ and the effective part of $T^k|_F$ has dimension $\geqslant 5$, and thus we apply (0.2.2) on F to draw the conclusions in Theorems A and B (note that $\pi_1(M) \cong \pi_1(F)$). If $n = 29$ (and thus $k \geqslant 5$), then we can get $\pi_1(M)$ is cyclic according to the proof of (0.2.3) for $n = 29$ in [15] (see the proof of Proposition 1.4 of [15]). If $23 \leqslant n \leqslant 27$, then we can conclude Theorems A and B by (0.2.2) and (0.2.3) (note that $k \geqslant 5$).

So far, it remains the case $\dim(F) \leqslant n - 8$ and $\dim(F) \geqslant 23$. For such case, we can finish the proof by applying inductive assumption on F (because $\pi_1(M) \cong \pi_1(F)$ and F admits an isometrical T^{k-1}-action with $k - 1 > \frac{\dim(F)+1}{8} + 1$). $\square$

2. The proof of Lemma 1.5

In the proof of Lemma 1.5, a basic tool is the following result.

Theorem 2.1. *(See [13].) Let M be a closed manifold of positive sectional curvature on which a torus T^1 acts by isometries, and let ϕ be an isometry on M which commutes with the T^1-action. Then ϕ preserves some T^1-orbit which is a circle.*

Note that, in Theorem 2.1, ϕ can act on the orbit space M/T^1 by isometry, and that ϕ has non-empty fixed point set on M/T^1. Then the following lemma may determine some topology properties of the fixed point set.

Lemma 2.2. *(See [6], [10, p. 63].) Let M be a closed Riemannian manifold on which T^1 acts by isometries. If ϕ is an isometry on $M^* = M/T^1$, then the Lefschetz number of ϕ is equal to the Euler characteristic of the fixed point set of ϕ, i.e. $\chi(F(\phi, M^*)) = Lef(\phi; M^*)$.*

In this paper, we denote by $F(G, X)$ the fixed point set of a group G-action on a topology space X. Recall that the Lefschetz number $\mathrm{Lef}(\phi; M^*) = \sum_i (-1)^i \mathrm{trace}(\phi_{*i})$, where $\mathrm{trace}(\phi_{*i})$ is the trace of the induced map by ϕ on $H_i(M^*; \mathbb{Q})$.

Next we will give the proof of Lemma 1.5 by assuming the following lemma, whose proof will be given afterwards.

Lemma 2.3. *For $i = 1$ or 2, in Lemma 1.1, if $\dim(F_i) = n - 4$ or $n - 6$ and if F_i is fixed by a T^1-subgroup, then either the universal covering space of M is a rational homology sphere or $\pi_1(M) \cong \pi_1(S)$, where S is a 3-dimensional space form of constant curvature one.*

Proof of Lemma 1.5. Let $\pi : \tilde{M} \to M$ be the Riemannian universal covering map. Assume that $\dim(F_1) = n - 4$ or $n - 6$, and that F_1 is fixed by the circle subgroup T^1. Note that we can lift the T^1-action to $\tilde{M}$ which commutes with the deck transformations, $\pi_1(M)$ [2, p. 63]. Obviously, $\tilde{F}_1 = \pi^{-1}(F_1)$, containing only one component by (1.2.3), is also fixed by T^1 in $\tilde{M}$, and thus $\pi_1(M)$ preserves $\tilde{F}_1$.

In the rest of the proof, we argue by contradiction. Assume that $\pi_1(M)$ contains a $\mathbb{Z}_p \oplus \mathbb{Z}_p$ subgroup with prime $p \neq 3$ and thus $\tilde{M}$ is a rational homology sphere (see Lemma 2.3). Since for $\dim(F_1) = n - 4$, the proof is an imitation (and it is easy to check that the conclusion holds for any prime p if $\dim(F_1) = n - 4$), we only give the proof for $\dim(F_1) = n - 6$.

Since $\tilde{M}$ is a rational homology sphere, the T^1-fixed point set $\tilde{F}_1$ is also a rational homology sphere (Smith theorem [2, p. 163]. It is easy to get $H^i(\tilde{M}, \tilde{F}_1; \mathbb{Q}) \cong \mathbb{Q}$ for $i = n-5, n$ and $H^i(\tilde{M}, \tilde{F}_1; \mathbb{Q}) = 0$ for $i \neq n-5, n$. By the Smith–Gysin sequence [2, p. 162]

$$\cdots \to H^i(\tilde{M}, \tilde{F}_1) \to H^{i-1}(\tilde{M}^*, \tilde{F}_1) \to H^{i+1}(\tilde{M}^*, \tilde{F}_1) \to H^{i+1}(\tilde{M}, \tilde{F}_1) \to \cdots,$$

we obtain that

$$H^i(\tilde{M}^*, \tilde{F}_1; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = n-5, n-3, n-1, \\ 0 & \text{other cases,} \end{cases}$$

where $\tilde{M}^* = \tilde{M}/T^1$.

Let α and β be the two generators of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Denote by $\hat{\alpha}$ the induced action of α on $\tilde{M}^*$. According to Theorem 2.1, $F(\hat{\alpha}, \tilde{M}^*)$, the fixed point set of $\hat{\alpha}$, is not empty. By Lemma 2.2

$$\chi\big(F(\hat{\alpha}, \tilde{M}^*, \tilde{F}_1)\big) = \mathrm{Lef}\big(\hat{\alpha}; \tilde{M}^*, \tilde{F}_1\big).$$

Note that α preserves the orientation[2] of $\tilde{M}$ and $\tilde{F}_1$. Thus by the definition of Lefschetz number, $\mathrm{Lef}(\hat{\alpha}; \tilde{M}^*, \tilde{F}_1) = 1$ or 3 (the case '1' occurs only when $p = 2$). Because $\hat{\alpha}$ has no fixed point on $\tilde{F}_1$, $\chi(F(\hat{\alpha}, \tilde{M}^*)) = \chi(F(\hat{\alpha}, \tilde{M}^*, \tilde{F}_1)) = 3$ or 1.

Let $P : \tilde{M} \to \tilde{M}^*$ be the quotient map. Note that for every component A of $F(\hat{\alpha}, \tilde{M}^*)$, $P^{-1}(A)$ is a fixed point component of $\alpha \circ t$ for some $t \in T^1$ with $\dim(P^{-1}(A)) = 1$, or 3, or 5. According to the Meyer's theorem (see Introduction and note that $P^{-1}(A)$ is totally geodesic), $H_1(P^{-1}(A); \mathbb{Q}) = 0$. By the above Smith–Gysin sequence, A is of positive Euler Characteristic. Moreover $\chi(A) \geqslant 3$ if $\dim(A) = 4$, and $\chi(A) = 2$ if $\dim(A) = 2$. It therefore follows that $F(\hat{\alpha}, \tilde{M}^*)$ consists of only one component of dimension 4 or 0, or one point and one component of dimension 2, or three points. In the first case, α preserves every T^1-orbit on $P^{-1}(A)$. Note that $\beta(F(\hat{\alpha}, \tilde{M}^*)) = F(\hat{\alpha}, \tilde{M}^*)$. By Theorem 2.1, β and α preserve one same T^1-orbit, and thus α and β generate a cyclic group (note that $\mathbb{Z}_p \oplus \mathbb{Z}_p = \langle \alpha, \beta \rangle$ acts on $\tilde{M}$ freely by isometries); a contradiction. In the latter two cases, $\hat{\alpha}$ and $\hat{\beta}$ fix one same point on $\tilde{M}^*$ (note that the last case needs $p \neq 3$), i.e., α and β preserve one same T^1-orbit; a contradiction again. □

In the rest of the section, we give the proof of Lemma 2.3. In the proof we will use the following two homology-determining results.

Lemma 2.4. *(See [18].) Let N^n be a closed oriented manifold, and let L^{n-k} be an embedded compact oriented submanifold without boundary. Suppose the inclusion $L \hookrightarrow N$ is $n-k-l$ connected and $n-k-2l > 0$. Let $[L] \in H_{n-k}(N; \mathbb{Z})$ be the image of the fundamental class of L in $H_*(N; \mathbb{Z})$, and let $e \in H^k(N; \mathbb{Z})$ be its Poincaré dual. Then the homomorphism*

$$\cup e : H^i(N; \mathbb{Z}) \to H^{i+k}(N; \mathbb{Z})$$

is surjective for $l \leqslant i < n-k-l$ and injective for $l < i \leqslant n-k-l$.

Lemma 2.5. *(See [18].) Let M be a simply connected closed n-manifold of positive sectional curvature. Assume that $\mathbb{Z}_p$ with p prime acts on M by isometries with a connected fixed point set N of codimension k. Then $H^i(M; \mathbb{Z}_p) = 0$ for $k \leqslant i \leqslant n-2k+1$.*

[2] The Weinstein's theorem [3] asserts that a closed positively curved manifold of odd dimension is orientable, and any orientation-reversing isometry on it has non-empty fixed point set.

Proof of Lemma 2.3. We first observe that, from the proof of Theorems A and B, if an involution in T^k has a fixed point component N of dimension $> \frac{n}{2}$ with the effective part of $T^k|_N$ having dimension $\geqslant k-1$, then we can assume that $\dim(N) = n-4$ or $n-6$, and the effective part of $T^k|_N$ has dimension equal to $k-1$. Another observation is that for $23 \leqslant n \leqslant 27$, the condition '$k > \frac{n+1}{8}+1$' satisfies (0.2.2) or (0.2.3), and thus we can give the proof only for $n \geqslant 29$.

Assume that F_i is fixed by circle subgroup T_i^1 for $i=1$ and 2, and let ι_i be the involution in T_i^1. Let $F = F_1 \cap F_2$, which is connected (see (1.2.3) and Remark 1.3) and of dimension $\geqslant \dim(F_1) + \dim(F_2) - n$. Let F_3 be a fixed point component of $\iota_1 \circ \iota_2$ with $F \subseteq F_3$. If $\dim(F) > \dim(F_1) + \dim(F_2) - n$, then by analyzing the isotropy representation of ι_1 and ι_2 on the normal space of F we can get that $\dim(F_i) - \dim(F) = 2$ for $i = 1$, or 2, or 3 (note that $\dim(F_1), \dim(F_2) = n-4$ or $n-6$), and thus $\pi_1(F_i)$ ($\cong \pi_1(M)$ by (1.2.1)) is cyclic (Lemma 1.4). Hence, without loss of generality, we assume that $\dim(F) = \dim(F_1) + \dim(F_2) - n$ (i.e. F_1 is transversal to F_2) and $\dim(F_1) \leqslant \dim(F_2)$. By (1.2.3), $F \hookrightarrow F_1$ is $\dim(F)$-connected.

Let $\pi : \tilde{M} \to M$ be the Riemannian universal covering map. By (1.2.3), $\tilde{F}_i = \pi^{-1}(F_i)$ and $\tilde{F} = \pi^{-1}(F) = \tilde{F}_1 \cap \tilde{F}_2$ are connected and thus simply connected by (1.2.1). Note that F is fixed by $T_2^1|_{F_1}$ in F_1, and that we can lift the T_2^1-action on F_1 to $\tilde{F}_1$ which commutes with $\pi_1(M)$ (see the beginning of the proof of Lemma 1.5). Obviously, $\tilde{F} \hookrightarrow \tilde{F}_1$ is also $\dim(F)$-connected, and $\tilde{F}$ is fixed by T_2^1 in $\tilde{F}_1$. By Lemma 2.4, for $\dim(F_1) - \dim(F) = 4$ (resp. 6), we have that

$$H^i(\tilde{F}_1;\mathbb{Q}) \cong H^{i+h}(\tilde{F}_1;\mathbb{Q}) \quad \text{for } 1 \leqslant i \leqslant \dim(F)-1 \text{ with } h = 4 \text{ (resp. 6)}. \tag{2.6}$$

We claim that $\tilde{F}_1$ is a rational homology sphere. Thus $\tilde{M}$ is a rational homology sphere because $\tilde{F}_1 \hookrightarrow \tilde{M}$ is at least $(n-11)$-connected by (1.2.1). Note that if $\dim(F_1) \equiv 1 \bmod 4$ and $\dim(F_1) - \dim(F) = 4$, then by (2.6) $H^4(\tilde{F}_1;\mathbb{Z}) \cong H^{\dim(F_1)-1}(\tilde{F}_1;\mathbb{Z}) = 0$ (and thus by Lemma 2.4 in turn $H^i(\tilde{F}_1;\mathbb{Z}) = 0$ for $1 \leqslant i \leqslant \dim(F_1)-1$ (i.e. $\tilde{F}_1$ is an integral homology sphere).

For $\dim(F_1) \equiv 3 \bmod 4$ or $\dim(F_1) - \dim(F) = 6$, we have the subclaim: $\tilde{F}$ is just the fixed point set of T_2^1 in $\tilde{F}_1$, otherwise $\pi_1(M) \cong \pi_1(S)$, where S is a 3-dimensional space form of constant curvature one. Note that the subclaim implies that there is prime p such that $\tilde{F} = F(\mathbb{Z}_p, \tilde{F}_1)$ where $\mathbb{Z}_p \subset T_2^1$. Hence by Lemma 2.5, for $\dim(F_1) - \dim(F) = 4$ (resp. 6)

$$H^i(\tilde{F}_1;\mathbb{Q}) = 0 \quad \text{for } h \leqslant i \leqslant \dim(F_1) - 2h + 1 \text{ with } h = 4 \text{ (resp. 6)}. \tag{2.7}$$

(2.7) and (2.6) together implies that $\tilde{F}_1$ is a rational homology sphere (note that $\dim(F_1) \geqslant n-6 \geqslant 23$).

In the rest of the proof, we only need to verify the subclaim.

Case 1. $\dim(F_1) - \dim(F) = 4$ and $\dim(F_1) \equiv 3 \bmod 4$. Assume that T_2^1 has other fixed point components F' in $\tilde{F}_1 \setminus \tilde{F}$. According to Remark 1.3, $\dim(F') = 1$. Since $\tilde{F} \hookrightarrow \tilde{F}_1$ is $\dim(F)$-connected, we have $H^j(\tilde{F};\mathbb{Q}) \cong H^j(\tilde{F}_1;\mathbb{Q})$ for $j \leqslant \dim(F)-1$. Moreover, by Lemma 2.4, it is easy to check (cf. (2.6)) that $H^{\dim(F_1)-j}(\tilde{F}_1;\mathbb{Q}) \cong 0$ (resp. $\cong \mathbb{Q}$ or 0) for $j = 1$ and 2 (resp. $j = 3$ and 4). Hence by the inequality (known as Borel–Smith theorem, p. 163 of [2])

$$\sum_i \operatorname{rank}\bigl(H^i(\tilde{F};\mathbb{Q})\bigr) + \sum_i \operatorname{rank}\bigl(H^i(F';\mathbb{Q})\bigr) = \sum_i \operatorname{rank}\bigl(H^i\bigl(F(T_2^1,\tilde{F}_1);\mathbb{Q}\bigr)\bigr) \leqslant \sum_i \operatorname{rank}\bigl(H^i(\tilde{F}_1;\mathbb{Q})\bigr),$$

we have $\sum_i \operatorname{rank}(H^i(F';\mathbb{Q})) \leqslant 2$ and thus F' is a circle (note that $\dim(F') = 1$). Since the T_2^1-action on $\tilde{F}_1$ commutes with $\pi_1(F_1)(\cong \pi_1(M))$, $\pi_1(M)$ preserves F' and thus $\pi_1(M)$ is cyclic.

Case 2. $\dim(F_1) - \dim(F) = 6$. Since $F_1 \cap F_2$ is connected, the subclaim is equivalent to "F_2 is just the fixed point set of T_2^1 in M, otherwise $\pi_1(M) \cong \pi_1(S)$", where S is a 3-dimensional space form of constant curvature one. Assume that T_2^1 has other fixed point components F' in $M \setminus F_2$. In this case, $\dim(F') \leqslant 3$.

From the above, we can assume that if an involution in T^k has a fixed point component N of dimension $> \frac{n}{2}$ and if the effective part of $T^k|_N$ has dimension $\geqslant k-1$, then we can assume that $\dim(N) = n-6$ and N is fixed by a T^1-subgroup. Moreover, every two of such fixed point components are transversal. Note that T^k preserves F', then there exists a circle T^k-orbit on F', i.e., F' contains a point x whose isotropy group contains a subgroup T^{k-1}. By (4.1.2) in Appendix A, we can find $k-2$ involutions $\iota_j \in T^{k-1}$ with fixed point components N_j of dimension $n-6$. On the other hand, the involution $\iota \in T_2^1$ has at least $n-5$ eigenvalue -1 on T_xM. Then $\iota \circ \iota_1 \circ \cdots \circ \iota_{k-2}$ has a fixed point component $N_{k-1} \ni x$ of dimension $\geqslant 6(k-2) - 4 + 1 > \frac{n}{2}$, so of dimension $n-6$ in fact.

Now we conclude that $E = \bigcap_{j=1}^{k-1} N_j \ni x$ is fixed by a subgroup T_1^{k-1}, and that E is connected (see Remark 1.3. Note that $\dim(E) = n - 6(k-1) \geqslant 1$). If $T_2^1 \subset T_1^{k-1}$, then $x \in E \subseteq F'$ and thus $\dim(E) \leqslant 3$. Repeating (1.2.3), we have

$$E \hookrightarrow \bigcap_{j=1}^{k-2} N_j \hookrightarrow \cdots \hookrightarrow N_1 \cap N_2 \hookrightarrow N_1 \text{ is } \dim(E)\text{-connected.}$$

It therefore follows that the subclaim is verified (see Remark 0.5). If $T_2^1 \not\subseteq T_1^{k-1}$, then there exists a T^k-fixed point component E' with $x \in E' \subseteq F'$, and thus $\dim(E') \leqslant 3$. Similar to the above, we can get (using (4.1.2) and Remark 4.2 in Appendix A) that E' is the intersection of k-many T^1-fixed point components of dimension $n-6$ which are transversal pairwisely, and $E' \hookrightarrow M$ is $\dim(E')$-connected and thus the subclaim is verified. $\square$

3. The proof of Lemma 1.6

In this section, we will give the proof of Lemma 1.6.

Proof of Lemma 1.6. As the proof of Lemma 2.3, we can only give the proof for $n \geqslant 29$.

Let $\pi : \tilde{M} \to M$ be the Riemannian universal covering map. We can lift the T^k-action on M to a $\tilde{T}^k$-action on $\tilde{M}$ which commutes with the deck transformations, $\pi_1(M)$ [2, p. 63]. Without loss of generality, we can let $\dim(F_1) = n-4$ or $n-6$, and thus $\tilde{M}$ is a rational homology sphere (Lemma 2.3). Let $\tilde{F}_1 = \pi^{-1}(F_1)$. Obviously, $\tilde{F}_1$, containing only one component by (1.2.3), is also fixed by a circle subgroup $T_0^1 \subset \tilde{T}^k$ on $\tilde{M}$, and thus $\pi_1(M)$ preserves $\tilde{F}_1$.

Since $\tilde{M}$ is a rational homology sphere, any T^l-fixed point set on $\tilde{M}$ is a rational homology sphere [2, p. 163], which is preserved by $\pi_1(M)$ (because $\tilde{T}^k$ commutes with $\pi_1(M)$). Let $\tilde{N}$ be any T^1-fixed point set on $\tilde{M}$ with the effective part of $\tilde{T}^k|_{\tilde{N}}$ having dimension $k-1$. Let $N = \pi(\tilde{N})$. Obviously, the effective part of $T^k|_N$ contains a T^{k-1} subgroup. On the other hand, $N \hookrightarrow M$ induced an onto map from $\pi_1(N)$ to $\pi_1(M)$ (because $\pi_1(M)$ preserves $\tilde{N}$). If $\dim(N) < \frac{n}{2}$, it is easy to check that $k-1 > \frac{\dim(N)+1}{4}$. According to (0.2.1), we get that $\pi_1(N)$ is cyclic, so is $\pi_1(M)$ cyclic. If $\dim(N) > \frac{n}{2}$, then $\pi_1(M) \cong \pi_1(N)$ (by (1.2.1)), and thus we can apply inductive assumption on N to draw the conclusion if $23 \leqslant \dim(N) \leqslant n-8$ in addition (note that $k-1 > \frac{\dim(N)+1}{8} + 1$ and $F_1 \cap N$ is a T^1-fixed point component of codimension $\leqslant 6$ in N), or apply (0.2.2) on N if $\frac{n}{2} < \dim(N) \leqslant 21$. Therefore, without loss of generality, we can assume that $\dim(\tilde{N}) = n-4$ or $n-6$.

a. We first give the proof for the conclusion (B2) in Theorem B according to the following two cases.

Case 1. The order of $\pi_1(M)$ is even.

Let α be a 2-order element in $\pi_1(M)$. By Theorem 2.1, α preserves some circle T_0^1-orbit out of $\tilde{F}_1$. In another word, there is a $t_0 \in T_0^1$ such that, out of $\tilde{F}_1$, $\alpha \circ t_0$ has a non-empty fixed point set $F(\alpha \circ t_0, \tilde{M})$. Note that we can assume that t_0 is of order 2^h for $h \geqslant 1$.

Let $\tilde{L}$ be a component of $F(\alpha \circ t_0, \tilde{M})$. Note that $\tilde{T}^k$ preserves $\tilde{L}$ because it commutes with $\pi_1(M)$, so there exists a circle $\tilde{T}^k$-orbit $\mathcal{O}$ on $\tilde{L}$ (see the argument after Lemma 1.1). By analyzing the representation of the isotropy subgroup of $\mathcal{O}$ containing a subgroup T_1^{k-1} on the normal space of $\mathcal{O}$, we can find $(k-1)$-many T_i^1-fixed point components, $\tilde{N}_i \supset \mathcal{O}$, of dimension $n-4$ or $n-6$ (see the second paragraph of the proof). Moreover, $\tilde{N}_i$ are transversal pairwisely. Since $\tilde{F}_1$ is the fixed point set of T_0^1, the isotropy subgroup of $\mathcal{O}$ is not $\tilde{T}^k$. This means that $\tilde{N}_i$ are the only T^1-fixed point components containing $\mathcal{O}$ of dimension $n-4$ or $n-6$. On the other hand, we can assume $\tilde{N}_i$ is one component of $F(t_i, \tilde{M})$ where $t_i \in T_i^1$ is of order 2. Otherwise $\dim(\tilde{N}') - \dim(\tilde{N}) = 2$ or $n - \dim(\tilde{N}') = 2$, where $\tilde{N}' \supset \tilde{N}$ is a component of $F(t_i, \tilde{M})$, and thus $\pi_1(M)$ is cyclic (Lemma 1.4).

Subcase 1. t_0 is of order 2.

Firstly, we assume that $\dim(F_1) = n-4$, and thus by (1.2.3) $\dim(\tilde{L}) \leqslant 3$. By analyzing the isotropy representation of $\alpha \circ t_0$ and $t_i \in T_i^1$ on the normal space of $\mathcal{O}$, one can check that $\alpha \circ t_0 \circ t_1 \circ \cdots \circ t_{k-1}$ has a fixed point component $\tilde{F} \supset \mathcal{O}$ of dimension $\geqslant 4(k-1) - 2 + 1 > \frac{n}{2}$ (note that $k > \frac{n+1}{8} + 1$ and $n \geqslant 23$), but $\tilde{F} \not\subseteq \tilde{N}_i$ for $1 \leqslant i \leqslant k-1$. Note that any involution t_i for $0 \leqslant i \leqslant k-1$ does not belong to $\pi_1(M)$, otherwise $\langle \alpha, t_i \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a subgroup of $\pi_1(M)$; a contradiction to Theorem A. Therefore, $\pi(\tilde{F})$ is fixed pointwisely by an involution $\iota \in T^k$ on M. Let N_k be a component of $F(\iota, M)$ with $N_k \supseteq \pi(\tilde{F})$. By (1.2.1), $\pi_1(M) \cong \pi_1(N_k)$ because $\dim(N_k) \geqslant \dim(\tilde{F}) > \frac{n}{2}$. Hence, without loss of generality, we can assume that $\dim(N_k) = n-4$ or $n-6$ and N_k is a T^1-fixed point component. This means that, in $\tilde{M}$, $\tilde{N}_k = \pi^{-1}(N_k)$ is another T^1-fixed point component containing $\mathcal{O}$ of dimension $n-4$ or $n-6$. Since $\tilde{N}_i$ $(\not\supseteq \tilde{F})$, $1 \leqslant i \leqslant k-1$, are the only T^1-fixed point components containing $\mathcal{O}$ of dimension $n-4$ or $n-6$, there must be $\tilde{N}_k = \tilde{M} = \tilde{F}$, which means that $\tilde{L} = \bigcap_{i=1}^{k-1} \tilde{N}_i$ and that $\alpha \circ t_0 \in T_1^{k-1}$. Note that $\tilde{L}$ is a component of $F(T_1^{k-1}, \tilde{M})$, and that $F(T_1^{k-1}, \tilde{M})$ is a rational homology sphere because $\tilde{M}$ is a rational homology sphere [2, p. 163]. Thus $\tilde{L}$ is just the rational homology sphere, $F(T_1^{k-1}, \tilde{M})$.

Secondly, we assume that $\dim(F_1) = \dim(\tilde{N}_i) = n-6$ (note that we can take the place of $\tilde{F}_1$ by $\tilde{N}_i$ for some $1 \leqslant i \leqslant k-1$), and thus by (1.2.3) $\dim(\tilde{L}) \leqslant 5$. Similarly, we can get $\tilde{N}_k \supset \mathcal{O}$ of dimension $\geqslant 6(k-1) - 4 + 1 > \frac{n}{2}$ and thus $\tilde{N}_k = \tilde{M}$, then $\tilde{L} = \bigcap_{i=1}^{k-1} \tilde{N}_i$ which is a rational homology sphere.

In any case, repeating (1.2.3), we get that $\tilde{L} \hookrightarrow \bigcap_{i=1}^{k-2} \tilde{N}_i \hookrightarrow \cdots \hookrightarrow \tilde{M}$ is $\dim(\tilde{L})$-connected. If $\dim(\tilde{L}) = 1$, then $\pi_1(M)$ is cyclic (note that $\pi_1(M)$ preserves $\tilde{L}$ because $\pi_1(M)$ commutes with $\tilde{T}^k$); if $\dim(\tilde{L}) = 3$ or 5, then $\pi_1(M) \cong \pi_1(L)$, where $L = \pi(\tilde{L})$. Thus the conclusion (B2) in Theorem B holds (see Remark 0.5).

Subcase 2. t_0 is of order 2^h with $h \geqslant 2$.

Since $\tilde{L}$ is fixed by $\alpha \circ t_0$, $\tilde{L}$ is also fixed by $t_0^{2^{h-1}}$ $(= (\alpha \circ t_0)^{2^{h-1}})$. Let $\tilde{E} \supseteq \tilde{L}$ be a component of $F(t_0^{2^{h-1}}, \tilde{M})$. Note that $\tilde{F}_1 \cap \tilde{E} = \emptyset$, and thus $\dim(\tilde{E}) \leqslant 5$ by (1.2.3). Let $\mathcal{O}'$ be a circle $\tilde{T}^k$-orbit on $\tilde{E}$, and let T_2^{k-1} be a subgroup of the isotropy group of $\mathcal{O}'$. Similar to Subcase 1, we can prove that $t_0^{2^{h-1}} \in T_2^{k-1}$ and that $\tilde{E} \hookrightarrow \tilde{M}$ is $\dim(\tilde{E})$-connected, afterwards we can draw the conclusion (B2) in Theorem B.

Case 2. The T^k-fixed point set is empty.

Since the T^k-fixed point set $F(T^k, M) = \emptyset$, the $\tilde{T}^k$-fixed point set $F(\tilde{T}^k, \tilde{M}) = \emptyset$. It is easy to check that we can find $T_0^2 \subset \tilde{T}^k$ such that $T_0^1 \subset T_0^2$ and $F(T_0^2, \tilde{M}) = \emptyset$ (because the $\tilde{T}^k$-orbit types on $\tilde{M}$ are finite). Since $\tilde{M}$ is a rational homology sphere, we have the following equality [2, p. 164]

$$n + 1 = \sum_H \bigl(\dim\bigl(F(H, \tilde{M})\bigr) + 1\bigr),$$

where H ranges over all $T^1 \subset T_0^2$ subgroups with non-empty fixed point set. It therefore follows that, out of $\tilde{F}_1$ (recall that $\tilde{F}_1$ is fixed by T_0^1), we can find another $T_1^1 \subset T_0^2$ subgroup with fixed point set $\tilde{K}$ of dimension $\leqslant 5$.

Note that $\tilde{K}$ is a rational homology sphere because $\tilde{M}$ is a rational homology sphere [2, p. 163]. On the other hand, $\pi_1(M)$ preserves $\tilde{K}$ (because $\pi_1(M)$ commutes with $\tilde{T}^k$), and thus $K \hookrightarrow M$ induces an onto map from $\pi_1(K)$ to $\pi_1(M)$, where $K = \pi(\tilde{K})$. If the effective part of $T^k|_K$ contains a torus of dimension 2, then $\pi_1(K)$ is cyclic by (0.2.1), so is $\pi_1(M)$ cyclic. Hence, without loss of generality, we can assume that $\tilde{K}$ is fixed by a T^{k-1}-subgroup in $\tilde{M}$, and that $\tilde{K}$ is the intersection of $(k-1)$-many T^1-fixed point sets of dimension $n-4$ or $n-6$. By the similar argument to prove "$\tilde{L} \hookrightarrow \tilde{M}$ is $\dim(\tilde{L})$-connected" in Case 1, we can obtain that $\tilde{K} \hookrightarrow \tilde{M}$ is $\dim(\tilde{K})$-connected, then we draw the conclusion (B2) in Theorem B.

b. The proof for the conclusion (B1) in Theorem B.

All notations in this part are same as those in Case 1 in part **a**.

If t_0 is of order 2, from Subcase 1 of Case 1 in part **a**, $\alpha \circ t_0 \circ t_1 \circ \cdots \circ t_{k-1}$ fixes $\tilde{M}$ pointwisely. In another word, α belongs to $\tilde{T}^k$, and thus α is in the center of $\pi_1(M)$ because $\tilde{T}^k$ and $\pi_1(M)$ commute.

If t_0 is of order 2^h with $h \geqslant 2$, from Subcase 2 of Case 1 in part **a**, we can get $t_0^{2^{h-1}} \in T_2^{k-1}$. Now we consider the isotropy representations of $t_0^{2^{h-2}}$ for $h \geqslant 3$ ($\alpha \circ t_0$ for $h = 2$) and T_2^{k-1} on the normal space of $\mathcal{O}'$. Note that there exists $t \in T_2^{k-1}$ such that $t^2 = t_0^{2^{h-1}}$. Because the representations of $t_0^{2^{h-2}}$ and t on the normal space of $\mathcal{O}'$ can be diagonalized simultaneously, $t_0^{2^{h-2}} \circ t$ or $t_0^{2^{h-2}} \circ t \circ t_0^{2^{h-1}}$ has fixed point set $\tilde{N}'_k$ of dimension $\geqslant \frac{n-1}{2} + 1 > \frac{n}{2}$. Similar to the proof for "$\tilde{N}_k = \tilde{M}$" in Subcase 1 of Case 1 in part **a**, we can obtain that $\tilde{N}'_k = \tilde{M}$ and thus $t_0^{2^{h-2}} \in T_2^{k-1}$ (note that $t, t_0^{2^{h-1}} \in T_2^{k-1}$). Step by step, we can prove that $t_0^{2^{h-3}}, \ldots, t_0^2$ and $\alpha \circ t_0$ all belong to T_2^{k-1}. It therefore follows that $\alpha \in \tilde{T}^k$, and thus α is in the center of $\pi_1(M)$. $\square$

Appendix A

As mentioned after Lemma 1.1, there exists a circle T^k-orbit $\mathcal{O}$ on M. Note that the isotropy group of $\mathcal{O}$ contains a T^{k-1} subgroup. By analyzing the isotropy representation of $\mathbb{Z}_2^{k-1} \subset T^{k-1}$ on the normal space of $\mathcal{O}$, we can find high dimensional fixed point sets by the following lemma (Refs. [6,14,15,18]).

Lemma 4.1. *(See [6].) Let $\mathbb{Z}_2^l \subset T^m \subset SO(2m)$ act on $\mathbb{R}^{2m}$ with $l > \frac{m+1}{4}$ and $m \geqslant 11$. Then*

(4.1.1) *$\mathbb{Z}_2^l$ contains at least two involutions with fixed point sets of dimension $\geqslant m$.*
(4.1.2) *In addition, if the fixed point set of every $\mathbb{Z}_2$-isotropy subgroup with dimension $\geqslant m$ actually has codimension 6, and there is no $\mathbb{Z}_2^2$-subgroup with fixed point set of codimension $\leqslant 10$, then there are at least $l-1$ many such $\mathbb{Z}_2$-isotropy subgroups.*

Proof. (4.1.1) Since the proof is an imitation of the corresponding conclusion for '$l > \frac{2m+1}{6}$' in [6], we only give the proof for $m = 14$ and $l = 4$.

Since $\mathbb{Z}_2^l \subset SO(2m)$, every element in $\mathbb{Z}_2^l$ can be represented by a square diagonal m-matrix with entries $\pm I_2$, where I_2 denotes the unit matrix of 2×2 form. For convenience, we use $A = (\pm 1, \cdots \pm 1)$ to represent such a diagonal matrix.

When $m = 14$ and $l = 4$, every element of $\mathbb{Z}_2^4$ has the following form,

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1).$$

One can easily find a $\mathbb{Z}_2^3$ subgroup which has the form

$$(1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1).$$

For any fixed jth position of the remaining 13 positions, there is a $\mathbb{Z}_2^2$ subgroup of $\mathbb{Z}_2^3$ whose jth position takes value 1. However, $\mathbb{Z}_2^3$ contains only 7 distinct $\mathbb{Z}_2^2$-subgroups, then there must be $\mathbb{Z}_2^2 \subset \mathbb{Z}_2^3$ whose elements are of form

$$(1, 1, 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1).$$

It is easy to see that $\mathbb{Z}_2^2$ has at least one involution A_1 with seven 1's, i.e., A_1 has fixed point set of dimension $\geqslant 14$. Assume that the ith position of A_1 is -1. Note that we can find a $\mathbb{Z}_2^3$ subgroup such that all of its elements have the ith position taking value 1. By the same argument we can find another involution A_2 with fixed point set of dimension $\geqslant 14$.

Now we give the idea of the proof of (4.1.1): to find a $\mathbb{Z}_2^3$ subgroup firstly such that it takes value 1 at $l-3$ positions, then to find a $\mathbb{Z}_2^2 \subset \mathbb{Z}_2^3$ subgroup which takes value 1 at two positions of the left $m-l+3$ positions, then to find an involution in $\mathbb{Z}_2^2$ with fixed point set of dimension $\geqslant m$.

(4.1.2) By (4.1.1), we have at least two involutions in $\mathbb{Z}_2^l$ with fixed point sets of codimension 6. Assume that we have found $l' \leqslant l-3$ such involutions, $A_1, \dots, A_{l'}$. We now show how to find another involution with fixed point set of codimension 6. Because there is no $\mathbb{Z}_2^2$-subgroup whose fixed point set has codimension $\leqslant 10$, each pair of $\{A_i\}$ satisfies that their positions of -1 are not overlapped. For convenience, we may assume that A_i take value -1 at the $(3i-2)$th, and $(3i-1)$th and $3i$th positions. Note that we can find a $\mathbb{Z}_2^3 \subset \mathbb{Z}_2^l$ subgroup which takes value 1 at the 1st, 4th, $\cdots$, $(3l'-2)$th positions, and the number of whose positions taking value 1 is at least $l-3$. It therefore follows that we can find another involution with fixed point sets of codimension 6 according to the idea in the proof of (4.1.1). Moreover, repeating the process above, we can find $l-2$ involutions, $A_1, \dots, A_{l-2}$, with fixed point sets of codimension 6, whose positions of -1 are not overlapped.

Next we will find the $(l-1)$th involution with fixed point set of codimension 6. Let $\mathbb{Z}_2^l$ be generated by $A_1, \dots, A_{l-2}$, α and β. Note that we can assume that α (otherwise β or $\alpha \circ \beta$) has $\frac{m-3(l-2)}{3}$ positions, in the latter $m-3(l-2)$ positions, taking value 1. Let A_{l-1} be the involution which is α composed with all A_i, where i $(1 \leqslant i \leqslant l-2)$ satisfies that α takes value 1 at 2 or 3 positions of the $(3i-2)$th, and $(3i-1)$th and $3i$th positions. It is easy to check that A_{l-1} has at least $\frac{m-3(l-2)}{3} + 2(l-2) \geqslant \frac{m}{2}$ positions taking value 1 (note that $m \geqslant 11$ and $l > \frac{m+1}{4}$). It therefore follows that, under the assumptions of (4.1.2), A_{l-1} is just the $(l-1)$th involution with fixed point set of codimension 6. $\square$

Remark 4.2. In (4.1.2), if $l > \frac{m+1}{4} + 1$, then we can find l involutions with fixed point sets of codimension 6. After finding $l-1$ involutions with fixed point sets of codimension 6, we can find the lth involution with fixed point set of codimension 6 using the similar argument to find A_{l-1} in the proof of (4.1.2) (but we cannot do the process only if $l > \frac{m+1}{4}$).

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